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BRST hamiltonian for bulk quantized gauge theory

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Abstract: By treating the bulk-quantized Yang-Mills theory as a constrained system we obtain a consistent gauge-fixed BRST hamiltonian in the minimal sector. This provides an independent derivation of the 5-d lagrangian bulk action. The ground state is independent of the (anti)ghosts and is interpreted as the solution of the Fokker-Planck equation, thus establishing a direct connection to the Fokker-Planck hamiltonian. The vacuum state correlators are shown to be in agreement with correlators in lagrangian 5-d formulation. It is verified that the complete propagators remain parabolic in one-loop dimensional regularization.

Keywords: Field Theories in Higher Dimensions, BRST Quantization, BRST Symmetry.

Contents

1. Introduction

The usual formulation [1] of 4-d gauge theory is based on the free (euclidean) lagrangian action

$$S = \frac{1}{4} \int_{\mathcal{M}^4} dx\, F_{\mu\nu} F^{\mu\nu} \tag{1.1}$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]} + [A_\mu, A_\nu]$ is the curvature of an SU(N) connection A. In recently introduced *bulk quantization* [2]–[5] (that arose from stochastic quantization approaches based on ideas of Parisi and Wu [6, 7]) one adds an extra fifth nonphysical dimension t to the spacetime 4-manifold $\mathcal{M}^4$. All the fields of the theory are then defined on the extended spacetime

$$\varphi(x)\,, \quad x \in \mathcal{M}^4 \qquad \longrightarrow \qquad \varphi(t,x)\,, \quad (t,x) \in \mathbf{R} \times \mathcal{M}^4$$

(This t corresponds to the stochastic evolution parameter or the Monte Carlo iteration time for numerical simulation). The connection $A_\mu dx^\mu$ is extended to include a fifth component $A_5 dt$ and one has $F_{5\mu} = \partial_5 A_\mu - \partial_\mu A_5 + [A_5, A_\mu]$.

A set of ghost fields is introduced with two independent $\mathbf{Z}$-gradings (ghost numbers) gh_s and gh_w corresponding to BRST differentials s and w, which raise the respective ghost numbers by one and satisfy

$$(s+w)^2 = 0 \qquad \Rightarrow \qquad s^2 = 0\,, \quad w^2 = 0\,, \quad sw = -ws \tag{1.2}$$

The operator w provides a BRST implementation of the 5-d gauge symmetry, analogous to the usual BRST operator [8, 9] (usually called s!) connected with Faddeev-Popov ghosts.

Its cohomology $H^0(w)$ defines observables. The operator s acts like a rigid supersymmetry operator and has trivial cohomology. Observables are *not* required to be s-exact. Fields with odd total ghost number $gh \equiv gh_s + gh_w$ anticommute. The action of s and w on the fields is defined as

$$
\begin{array}{rrrr}
sA_\mu = \Psi_\mu & s\Psi_\mu = 0 & s\bar{\Psi}_\mu = \Pi_\mu & s\Pi_\mu = 0 \\
sA_5 = \Psi_5 & s\Psi_5 = 0 & s\bar{\Psi}_5 = \Pi_5 & s\Pi_5 = 0 \\
sc = \Phi & s\Phi = 0 & s\bar{\Phi} = \bar{c} & s\bar{c} = 0 \\
s\lambda = \mu & s\mu = 0 & s\bar{\mu} = \bar{\lambda} & s\bar{\lambda} = 0
\end{array}
$$

$$
\begin{array}{llll}
wA_\nu = D_\nu\lambda & w\Psi_\nu = -[\lambda,\Psi_\nu] - D_\nu\mu & w\bar{\Psi}_\nu = -[\lambda,\bar{\Psi}_\nu] & w\Pi_\nu = -[\lambda,\Pi_\nu] + [\mu,\bar{\Psi}_\nu] \\
wA_5 = D_5\lambda & w\Psi_5 = -[\lambda,\Psi_5] - D_5\mu & w\bar{\Psi}_5 = -[\lambda,\bar{\Psi}_5] & w\Pi_5 = -[\lambda,\Pi_5] + [\mu,\bar{\Psi}_5] \\
wc = -[\lambda,c] - \mu & w\Phi = -[\lambda,\Phi] + [\mu,c] & w\bar{\Phi} = -[\lambda,\bar{\Phi}] & w\bar{c} = -[\lambda,\bar{c},] + [\mu,\bar{\Phi}] \\
w\lambda = -\frac{1}{2}[\lambda,\lambda] & w\mu = -[\lambda,\mu] & w\bar{\mu} = -[\lambda,\bar{\mu}] + \bar{\Psi}_5 & w\bar{\lambda} = -[\lambda,\bar{\lambda}] + [\mu,\bar{\mu}] + \Pi_5 \,.
\end{array}
\tag{1.3}
$$

Here $D_\mu = \partial_\mu + [A_\mu,\,]$ and $D_5 = \partial_5 + [A_5,\,]$ denotes the usual gauge covariant derivative. With some obvious renaming of fields this is the BRST algebra of [5], with a minor *exception*. To make the action of w on the quartet $A_5, \Psi_5, \bar{\Psi}_5, \Pi_5$ symmetric in form, as it now is, to the action of w on $A_\mu, \Psi_\mu, \bar{\Psi}_\mu, \Pi_\mu$ we made the field redefinitions

$$\bar{\Psi}_5 \equiv \bar{m} + [\lambda,\bar{\mu}] \quad \text{and} \quad \Pi_5 \equiv -l - \left[\lambda,\bar{\lambda}\right] + [\mu,\bar{\mu}]\,. \tag{1.4}$$

Otherwise one has in [5]

$$
\begin{array}{lr}
s\bar{m} = l & sl = 0 \\
w\bar{\lambda} = -l & wl = 0 \\
w\bar{\mu} = \bar{m} & w\bar{m} = 0
\end{array}
$$

which, although simpler, lacks the aforementioned symmetry and moreover leads to more cubic ghost interaction terms in $I_{\rm gf}$ than our choice here.

The 5-d action for the theory is s-exact and w-closed

$$wI = 0 \tag{1.5}$$

and is given by

$$
\begin{aligned}
I &= I_0 + I_{\rm gf} \\
I_0 &\equiv \int d^5x s\Big[\bar{\Psi}_\mu\left(F^{5\mu} - D_\lambda F^{\lambda\mu} + \Pi^\mu + \left[\bar{\Psi}^\mu, c\right]\right) + \bar{\Phi}\left(\Psi_5 - a'^{-1}D_\mu\Psi^\mu - \left(D_5 - a'^{-1}D^2\right)c\right)\Big] \\
I_{\rm gf} &\equiv \int d^5x\, ws\left[\bar{\mu}\left(A_5 - a^{-1}\partial\cdot A\right)\right]
\end{aligned}
\tag{1.6}
$$

where a and a' are positive constant parameters. After expansion, the w-exact piece $I_{\rm gf}$ fixes the gauge for A_μ and Ψ_μ to $A_5 = a^{-1}\partial\cdot A$ and $\Psi_5 = a^{-1}\partial\cdot\Psi$. The theory is well-defined in this gauge and one has convergence of longitudinal modes. From the 4-d point of view this axial type 5-d gauge condition actually corresponds to an infinitesimal *gauge transformation* $\delta A_\mu = D_\mu a^{-1} \partial \cdot A$, so there is no Gribov obstruction associated with gauge fixing (see [5]).

Because all free ghost propagators are retarded, closed ghost loops vanish (except for tadpoles which can be ignored). Since ghost number is conserved, as long as one doesn't compute ghost correlators the effect of integrating out the ghosts is simply to suppress the ghosts in the action which, after integrating out Π_μ as well and rescaling t, yields

$$I'_{\text{red}} = -\frac{1}{4} \int d^5x \left[a^{-1} \left(\partial_t A_\mu - D_\mu \partial \cdot A \right)^2 + a \left(D_\lambda F^{\lambda\mu} \right)^2 \right]. \tag{1.7}$$

After analyzing this action in the Landau gauge limit $a \searrow 0$ one finds that the weight is concentrated in the Gribov region, i.e., where $\partial \cdot A = 0$ and the Faddeev operator is positive, $-\partial \cdot D(A) \geqslant 0$. The physical content of the 4-d theory, such as correlators, is recovered by going to a time slice $t =$ constant. The reader is referred to [5] for details.

We will address here the question of finding the proper hamiltonian corresponding to (1.6). An outline of how we proceed is as follows. We consider just I_0, the gauge non-fixed part of the action, and read off the hamiltonian, which has simple first class constraints. One has a choice of whether or not to include $A_5, \Psi_5, \bar{\Psi}_5, \Pi_5$ among the canonical variables; the phase space without these variables is called the the *minimal sector*. In the hamiltonian formalism the first class constraints are generators of gauge transformations, and hence of w. To quantize the system one needs a BRST gauge-fixed hamiltonian. According to homological BRST theory, a ghost-antighost pair is introduced for each constraint and used to construct a BRST generator Ω for w, which we choose to do in the minimal sector for reasons outlined below. We then obtain a gauge-fixed hamiltonian $H^{\min} = H_C^{\min} - \{\Omega, K\}$, the gauge being fixed by the second term with K chosen so as to give action $I^{\min}$ (a reduced form of I that results after integrating out non-minimal fields).

We then go on to show that the complete ghost propagators remain retarded in one-loop dimensional regularization. The retarded character of the full ghost propagators allows us to establish an equivalence between the quantum hamiltonian and lagrangian correlation functions. We also argue that the ground state wave function P has trivial ghost dependence, which provides a direct connection to the Fokker-Planck equation

$$-\int d^4x \, \frac{\delta}{\delta A_\mu(x)} \left[\frac{\delta}{\delta A^\mu(x)} - \frac{\delta S_{\text{YM}}}{\delta A^\mu(x)} + a^{-1} D_\mu \partial \cdot A(x) \right] P(A) = 0\,. \tag{1.8}$$

2. Constrained hamiltonian

The gauge non-fixed part of the action after expansion is

$$\begin{aligned}
I_0 &= I_F + I_\Pi + I_c \\
I_F &= \int d^5x \left[\Pi_\mu (F^{5\mu} - D_\lambda F^{\lambda\mu}) - \bar{\Psi}_\mu \left(D^{[5}\Psi^{\mu]} - D_\lambda D^{[\lambda}\Psi^{\mu]} - [F^{\mu\nu}, \Psi_\nu] \right) \right] \\
I_\Pi &= \int d^5x \left[\Pi^2 + 2\Pi_\mu \left[\bar{\Psi}^\mu, c \right] + \left[\bar{\Psi}^\mu, \bar{\Psi}_\mu \right] \Phi \right]
\end{aligned}$$

$$I_c = \int d^5x \bar{c}\left[(\Psi_5 - a'^{-1}D_\mu\Psi^\mu - (D_5 - a'^{-1}D^2)c\right] \qquad (2.1)$$
$$+ \int d^5x\, \bar{\Phi}\left[-\left(D_5 - a'^{-1}D^2\right)\Phi - \left[\Psi_5 - a'^{-1}D_\mu\Psi^\mu, c\right] + a'^{-1}\left[\Psi^\mu, 2D_\mu c - \Psi_\mu\right]\right] .$$

The gauge fixing term for future reference is

$$I_{\rm gf} = \int d^5x \Big[\Pi_5\left(A^5 - a^{-1}\partial\cdot A\right) + \bar{\Psi}_5\left(\Psi^5 - a^{-1}\partial_\mu\Psi^\mu\right) - \bar{\lambda}\left(\partial_5 - a^{-1}D\cdot\partial\right)\lambda -$$
$$- \bar{\mu}\left(\left(\partial_5 - a^{-1}D\cdot\partial\right)\mu - a^{-1}\left[\Psi^\mu, \partial_\mu\lambda\right]\right)\Big] . \qquad (2.2)$$

First we look at the equations of motion generated by varying I_0 with respect to the fields Ψ_5 and A_5

$$0 = \frac{\delta I_0}{\delta\Psi_5} = D_\mu\bar{\Psi}^\mu + [\bar{\Phi}, c] - \bar{c} \equiv \varphi_1 ,$$
$$0 = \frac{\delta I_0}{\delta A_5} = D_\mu\Pi^\mu + [\bar{\Psi}_\mu, \Psi^\mu] + [\bar{c}, c] + [\bar{\Phi}, \Phi] \equiv \varphi_2 \qquad (2.3)$$

and obtain what are called *primary constraints* φ_1 and φ_2. Note that we use the usual convention that all functional derivatives with respect to Grassman fields are *left* derivatives. It is of interest to observe that the constraints satisfy

$$s\varphi_1 = \varphi_2 \qquad (2.4)$$

We substitute these constraints into the action I_0, and obtain a reduced form of the action $I_0^{\rm min}$, where all terms linear in Ψ_5 and A_5 have been eliminated by the equations of motion (2.3). In this approach Ψ_5 and A_5 play the role of lagrange multipliers and are *not* canonical variables. This is analogous to the role A_0 plays in enforcing Gauss law $D_jE^j = 0$ in the minimal hamiltonian for electromagnetism or Yang-Mills, where only A_j and E_j are treated as canonical variables. Thus

$$I_0 = \int d^5x\left(\dot{A}_\mu\Pi^\mu + \dot{\Psi}_\mu\bar{\Psi}^\mu + \dot{c}\bar{c} - \dot{\Phi}\bar{\Phi} - \mathcal{H}_C^{\rm min} - \Psi^5\varphi_1 - A^5\varphi_2\right) \qquad (2.5)$$

and the reduced form of the action is

$$I_0^{\rm min} = \int d^5x\left(\dot{A}_\mu\Pi^\mu + \dot{\Psi}_\mu\bar{\Psi}^\mu + \dot{c}\bar{c} - \dot{\Phi}\bar{\Phi} - \mathcal{H}_C^{\rm min}\right), \qquad (2.6)$$
$$-\mathcal{H}_C^{\rm min} = \Pi^2 + \left(2[\bar{\Psi}_\mu, c] - D_\lambda F^{\lambda\mu}\right)\Pi_\mu + \bar{\Psi}_\mu\left(D_\lambda D^{[\lambda}\Psi^{\mu]} + [F^{\mu\nu}, \Psi_\nu]\right) + [\bar{\Psi}_\mu, \bar{\Psi}^\mu]\Phi +$$
$$+a'^{-1}\bar{c}(D^2c - D_\mu\Psi^\mu) + a'^{-1}\bar{\Phi}\left(D^2\Phi + [D_\mu\Psi^\mu, c] + [\Psi^\mu, 2D_\mu c - \Psi_\mu]\right) . \qquad (2.7)$$

One reads off $H_C^{\rm min}$ from I_0 by dropping kinetic terms and setting $\varphi_1 = \varphi_2 = 0$. Since I_0, as given in (1.6), is s-exact it follows that $H_C^{\rm min}$ is also s-exact and can be expressed as

$$H_C^{\rm min} = \int d^4x\, s\left[\bar{\Psi}_\mu\left(\Pi^\mu - D_\lambda F^{\lambda\mu} - [\bar{\Psi}^\mu, c]\right) - a'^{-1}\bar{\Phi}D_\mu\left(\Psi^\mu - D^\mu c\right)\right] . \qquad (2.8)$$

We now proceed with the analysis of this constrained gauge system which goes according to a standard prescription, as follows. The reader is referred to to [10, 11] for background on constrained systems. The constrained hamiltonian is written

$$H^{\min} \equiv H_C^{\min} + \int d^4x\, u^j \varphi_j\,, \qquad j = 1,2 \tag{2.9}$$

where the u^j are the lagrange multipliers, here Ψ_5 and Π_5, enforcing the primary constraints φ_j. In general the u^j may be chosen to be included in the canonical variables. In our case this would correspond to including the $A_5, \Psi_5, \bar{\Psi}_5, \Pi_5$ quartet in the phase space. Such an approach is termed *nonminimal*. Our analysis will be based on choosing the more economical phase space, hence we use the $H^{\min}$ notation. One can in principle consider a nonminimal treatment, but it is inconvenient for this system and hence remarks related thereto are relegated to appendix A.

We use $H_C^{\min}$ to denote the canonical hamiltonian with the corresponding action $I_0^{\min}$. The action I_0 with constraints is then called the *extended action* and $H^{\min}$ is termed *extended hamiltonian*. The $\approx$ notation is introduced to represent *weak* equality, that is equality modulo functions that vanish on the constraint surface in phase space described by $\varphi_j = 0$. $H^{\min}$ then determines time evolution of all functions F of the fields by

$$\dot{F} \approx -\{H_C^{\min} + u^j\varphi_j, F\} \tag{2.10}$$

Here $\{\,,\,\}$ is the graded Poisson bracket. It is defined on functions F and G of the fields as

$$\begin{aligned}\{F(x), G(y)\} = \int d^4z \Bigg[&\frac{\delta F(x)}{\delta\varphi^a(z)}\frac{\delta G(y)}{\delta p_a(z)} - \frac{\delta F(x)}{\delta p_a(z)}\frac{\delta G(y)}{\delta\varphi^a(z)} + \\ &+ (-)^{gh(F)}\left(\frac{\delta F(x)}{\delta\theta^b(z)}\frac{\delta G(y)}{\delta\pi_b(z)} + \frac{\delta F(x)}{\delta\pi_b(z)}\frac{\delta G(y)}{\delta\theta^b(z)}\right)\Bigg]\end{aligned} \tag{2.11}$$

where φ^a denote all the commuting fields, the p_a their momenta, and similarly for the odd θ^b and π_b. It satisfies a graded Jacobi identity

$$(-)^{gh(F_3)gh(F_1)}\{F_1, \{F_2, F_3\}\} + \text{cyclic perms} = 0\,. \tag{2.12}$$

Note that one has for odd fields $\{\theta(x), \pi(y)\} = \{\pi(x), \theta(y)\} = -\delta(x-y)$. Moreover, if one expresses s in terms of functional derivatives

$$s = \int d^4x \left[\Psi^\mu(x)\frac{\delta}{\delta A^\mu(x)} + \cdots\right] \tag{2.13}$$

one finds that s acts as a graded derivation with respect to the bracket

$$s\{F,G\} = \{sF, G\} + (-)^{gh(F)}\{F, sG\}\,. \tag{2.14}$$

The constraints φ_j must be preserved in time, so we apply (2.10) to φ_j and get $\{H_C^{\min}, \varphi_j\} \approx 0$ which generates no further (what would be termed *secondary*) constraints. Some computation (the Jacobi identity is useful) shows that the constraints φ_m close to generate a Lie algebra (the structure functions are all constant)

$$\begin{aligned}\{\varphi_1^a(x), \varphi_2^b(y)\} &= \delta(x-y) f^{ab}{}_c \varphi_1^c(y) \\ \{\varphi_2^a(x), \varphi_2^b(y)\} &= \delta(x-y) f^{ab}{}_c \varphi_2^c(y)\,.\end{aligned} \tag{2.15}$$

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Here the $f^{ab}{}_c$ are the (totally antisymmetric) structure constants of $\mathfrak{su}(\mathrm{N})$. Note that we expect the second relation to follow from the first by way of (2.14). It is not hard to check that $\{H_C^{\min}, \varphi_1\} = \{H_C^{\min}, \varphi_2\} = 0$. Thus there are no secondary constraints at all. Moreover, the constraints are *irreducible*, meaning that the equations $\varphi_m = 0$ are independent.

For constrained hamiltonian systems a functional F whose bracket with every constraint (including secondary, if they are present) vanishes weakly

$$\{\varphi_m, F\} \approx 0 \tag{2.16}$$

is said to be *first class*. First class functionals have important properties. From the Jacobi identity it follows that the bracket of first class functions is first class. Also, the first class constraints are the generators of gauge transformations [11]

$$\delta_\epsilon F(x) = -\int d^4y\, \epsilon^m(y)\{\varphi_m(y), F(x)\}\,. \tag{2.17}$$

The ghost numbers of the infinitesimal gauge parameters ϵ^m are chosen so as to leave the ghost number of F invariant, i.e., $gh(\epsilon^m) = -gh(\varphi_m)$. In our case $gh(\epsilon_1) = 1$ and $gh(\epsilon_2) = 0$. Since all constraints are first class, one need not introduce Dirac brackets and the analysis of the system is considerably simplified. We remark that the analysis of the constraints is in general highly dependent on where one draws the minimal sector, so that constraints that are first class in one treatment may be second class in another, likewise regarding primary and secondary, and some constraints may be altogether absent.

Now, for each generator φ_m, the corresponding gauge transformation is given by (2.17), but without sum on m. One has

$$\begin{aligned}
\delta_{\epsilon_2} A^a_\mu &= -(D_\mu\epsilon_2)^a & \delta_{\epsilon_1} A^a_\mu &= 0\\
\delta_{\epsilon_2} \Psi^a_\mu &= [\epsilon_2, \Psi_\mu]^a & \delta_{\epsilon_1} \Psi^a_\mu &= -(D_\mu\epsilon_1)^a\\
\delta_{\epsilon_2} \bar\Psi^a_\mu &= [\epsilon_2, \bar\Psi_\mu]^a & \delta_{\epsilon_1} \bar\Psi^a_\mu &= 0\\
\delta_{\epsilon_2} \Pi^a_\mu &= [\epsilon_2, \Pi_\mu]^a & \delta_{\epsilon_1} \Pi^a_\mu &= [\epsilon_1, \bar\Psi_\mu]^a\\
\delta_{\epsilon_2} c^a &= [\epsilon_2, c]^a & \delta_{\epsilon_1} c^a &= -\epsilon_1^a\\
\delta_{\epsilon_2} \Phi^a &= [\epsilon_2, \Phi]^a & \delta_{\epsilon_1} \Phi^a &= [\epsilon_1, c]^a\\
\delta_{\epsilon_2} \bar\Phi^a &= [\epsilon_2, \bar\Phi]^a & \delta_{\epsilon_1} \bar\Phi^a &= 0\\
\delta_{\epsilon_2} \bar c^a &= [\epsilon_2, \bar c]^a & \delta_{\epsilon_1} \bar c^a &= [\epsilon_1, \bar\Phi]^a
\end{aligned} \tag{2.18}$$

The full gauge transformations (2.17) of the fields are given by

$$\begin{aligned}
\delta A_\nu &= -D_\nu\epsilon_2 & \delta\Psi_\nu &= [\epsilon_2, \Psi_\nu] - D_\nu\epsilon_1 & \delta\bar\Psi_\nu &=[\epsilon_2, \bar\Psi_\nu] & \delta\Pi_\nu &=[\epsilon_2, \Pi_\nu] + [\epsilon_1, \bar\Psi_\nu]\\
\delta c &=[\epsilon_2, c] - \epsilon_1 & \delta\Phi &= [\epsilon_2, \Phi] + [\epsilon_1, c] & \delta\bar\Phi &=[\epsilon_2, \bar\Phi] & \delta\bar c &=[\epsilon_2, \bar c,] + [\epsilon_1, \bar\Phi]
\end{aligned} \tag{2.19}$$

which coincides with the remnant (after the A_5 quartet is gone) of the w algebra (1.3) when the ϵ parameters are replaced by variables of opposite statistics

$$\epsilon_1 \longrightarrow \mu\,, \qquad \epsilon_2 \longrightarrow -\lambda\,, \qquad s\lambda = \mu \tag{2.20}$$

with $gh_s(\lambda) = 0$, $gh_w(\lambda) = 1$ and $gh_w(\mu) = gh_s(\mu) = 1$. In [5] transformation properties were *imposed* on the fields by hand, as each new field was added to the action, in such a way as to have w invariance of the action I. From our point of view the gauge algebra and field transformations are in fact *determined* by the constraints, i.e., the action.

As for the lagrange multiplier fields, one has the freedom of assigning to them any gauge transformation properties one sees fit, and we may therefore choose to transform them in such a way as to make the entire action

$$I_0 = I_0^{\min} - \int d^5x \, (\Psi_5 \varphi_1 + A_5 \varphi_2) \tag{2.21}$$

gauge invariant. This can always be arranged even in the most general cases with second class constraints [11], and in our case amounts to (not surprisingly) setting

$$\delta\Psi_5 = -D_5\epsilon_1 + [\epsilon_2, \Psi_5], \qquad \delta A_5 = -D_5\epsilon_2 \,. \tag{2.22}$$

3. Minimal BRST hamiltonian

The following considerations are direct consequences of standard results of homological BRST theory (we refer to [11] for details). The extended phase space is introduced by including in the minimal sector a ghost-antighost conjugate pair for each of the constraints φ_1 and φ_2. Hence we add $\mu, \bar{\mu}$ for φ_1 and $\lambda, \bar{\lambda}$ for φ_2, and as the notation indicates, identify them with the ghost fields in $I_{\rm gf}$. The corresponding kinetic terms $-\dot{\mu}\bar{\mu} + \dot{\lambda}\bar{\lambda}$ are included in the action. Note the minus sign in $\{\mu(x), \bar{\mu}(y)\} = -\delta(x-y)$.

By inspection of (1.3), one easily finds the generator Q for s on the extended phase space

$$s = -\{Q, \,\}\,, \qquad Q = \int d^4x \, \left(\Psi_\mu \Pi^\mu + \Phi\bar{c} + \mu\bar{\lambda}\right) \tag{3.1}$$

Therefore we have

$$H_C^{\min} = -\{Q, X\}\,, \qquad X = -\int d^4x \, \left[\bar{\Psi}_\mu \left(\Pi^\mu - D_\lambda F^{\lambda\mu} + [\bar{\Psi}^\mu, c]\right) - a'^{-1}\bar{\Phi} D_\mu \left(\Psi^\mu - D^\mu c\right)\right] \tag{3.2}$$

A main theorem of BRST theory [11] provides the existence of a BRST generator Ω for w which, because the φ_j generate a genuine Lie algebra, takes a particularly simple form

$$\begin{aligned} w = -\{\Omega, \,\}, \qquad \Omega &= \int d^4x \left(\mu\varphi_1 - \lambda\varphi_2 - \frac{1}{2}\bar{\lambda}[\lambda, \lambda] - \bar{\mu}[\mu, \lambda]\right) \\ &= -\left\{Q, \int d^4x \left(\lambda\varphi_1 - \frac{1}{2}\bar{\mu}[\lambda, \lambda]\right)\right\} \end{aligned} \tag{3.3}$$

Thus Ω is Q-exact which implies $\{Q, \Omega\} = 0$, as expected. We remark that in theories where the constraints do not generate a closed algebra with constant structure functions, the BRST generator may be much more complicated (an infinite series expansion in a ghost degree). So again, we see the attractive simplicity of this 5-d theory.

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One needs to construct an appropriate BRST invariant extension of $H_C^{\rm min}$ and then gauge-fix it. But $H_C^{\rm min}$ is already w-invariant so the gauge-fixed BRST hamiltonian corresponding to $I = I_0 + I_{\rm gf}$, as expressed in terms of this minimal set of fields, is then simply given by

$$H^{\rm min} = H_C^{\rm min} + H_{\rm gf}^{\rm min} = -\{Q, X\} - \{\Omega, K\} \tag{3.4}$$

where the *gauge fixing fermion* K, as it is frequently called, is chosen to be

$$K = -\{Q, f\} \equiv -\left\{Q, a^{-1}\int d^4x\, \bar\mu \partial\cdot A\right\} = a^{-1}\int d^4x\, \left(\bar\lambda\partial\cdot A + \bar\mu\partial\cdot\Psi\right). \tag{3.5}$$

From the Jacobi identity and $\{\Omega, Q\} = 0$ one has

$$\{\Omega, K\} = \{Q, \{\Omega, f\}\} \tag{3.6}$$

so $H^{\rm min}$ can be written in the compact form

$$H^{\rm min} = -\{Q, \bar Q\}, \qquad \bar Q = X + \{\Omega, f\}\,. \tag{3.7}$$

Explicitly one has

$$-\{\Omega, K\} = a^{-1}\int d^4x\, \left(\varphi_1\partial\cdot\Psi - \varphi_2\partial\cdot A - \bar\mu D\cdot\partial\mu - \bar\lambda D\cdot\partial\lambda - \bar\mu[\bar\Psi^\mu, \partial_\mu\lambda]\right). \tag{3.8}$$

Thus the BRST hamiltonian in its fully expanded form is

$$\begin{aligned} H^{\rm min} &= -\{Q, \bar Q\} \\ &= -\{Q, X\} - \{\Omega, K\} \\ &= \int d^4x - \Big(\Pi^2 + \Pi_\mu\big(2[\bar\Psi_\mu, c] - D_\lambda F^{\lambda\mu}\big) + \bar\Psi_\mu\big(D_\lambda D^{[\lambda}\Psi^{\mu]} + [F^{\mu\nu}, \Psi_\nu]\big) + [\bar\Psi_\mu, \bar\Psi^\mu]\Phi \\ &\qquad\qquad + \frac{1}{a'}\big[\bar c(D^2c - D_\mu\Psi^\mu) + \bar\Phi\big(D^2\Phi + [D_\mu\Psi^\mu, c] + [\Psi^\mu, 2D_\mu c - \Psi_\mu]\big)\big]\Big) + \\ &\quad + \frac{1}{a}\Big((D_\mu\bar\Psi^\mu + [\bar\Phi, c] - \bar c)\partial\cdot\Psi - (D_\mu\Pi^\mu + [\bar\Psi_\mu, \Psi^\mu] + [\bar c, c] + [\bar\Phi, \Phi])\partial\cdot A - \\ &\qquad - \bar\mu D\cdot\partial\mu - \bar\lambda D\cdot\partial\lambda - \bar\mu[\bar\Psi^\mu, \partial_\mu\lambda]\Big)\,. \end{aligned} \tag{3.9}$$

Our canonical treatment agrees with [5] because one can easily check that after integrating out $\Pi_5, A_5, \bar\Psi_5, \Psi_5$ in the lagrangian action I [= (2.1) + (2.2)] one gets precisely

$$I^{\rm min} = \int d^5x\, \left(\dot A_\mu\Pi^\mu + \dot\Psi_\mu\bar\Psi^\mu + \dot c\bar c - \dot\Phi\bar\Phi + \dot\lambda\bar\lambda - \dot\mu\bar\mu - \mathcal{H}^{\rm min}\right) \tag{3.10}$$

So in fact, what we have done here is give a consistent constructive derivation of the (reduced form of) action I based on the canonical analysis of the constrained hamiltonian.

4. Propagators

In this section we study the propagators and show that all complete ghost (and ghost of ghost) propagators stay retarded in one-loop dimensional regularization. This is an important feature of bulk quantization and will be key to establishing the advertised results on correlators ant the ground state in the next section..

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Let us then begin by first computing the *free* propagators by inverting the quadratic part of the action $I^{\rm min}$, which is given by the quadratic form

$$I^{\rm min}_{(0)} = -\int d^5x\, A^\mu \left(-\delta_{\mu\nu}\partial_5 + a^{-1}\partial_\mu\partial_\nu + \Box^{\rm tr}_{\mu\nu}\right)\Pi^\nu - \Pi^2 + \\ +\bar\Phi\left(\partial_5 - {a'}^{-1}\Box\right)\Phi + \bar\lambda\left(\partial_5 - {a'}^{-1}\Box\right)\lambda + \bar\mu\left(\partial_5 - {a'}^{-1}\Box\right) + \\ +(\bar\Psi_\mu, \bar c)\begin{pmatrix}\delta_{\mu\nu}\partial_5 - a^{-1}\partial_\mu\partial_\nu - \Box^{\rm tr}_{\mu\nu} & 0 \\ (a-a')\partial_\nu/aa' & \partial_5 - {a'}^{-1}\Box\end{pmatrix}\begin{pmatrix}\Psi_\nu \\ c\end{pmatrix}. \tag{4.1}$$

Here

$$\Box^{\rm tr}_{\mu\nu} = \Box P^{\rm tr} = \delta_{\mu\nu}\Box - \partial_\mu\partial_\nu \tag{4.2}$$

and $P^{\rm tr}$ and $P^{\rm lg}$ are the usual transverse and longitudinal projectors. They may be defined via their Fourier transforms (denoted by $\hat{}$)

$$\hat P^{\rm tr}_{\mu\nu}(p) = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \qquad \hat P^{\rm lg}_{\mu\nu}(p) = \hat P^{\rm tr\perp}_{\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2}. \tag{4.3}$$

and provide an orthogonal decomposition $\mathbf{1} = P^{\rm tr}(\partial) + P^{\rm lg}(\partial)$.

The single blocks are trivial to invert and give the free momentum space propagators

$$\hat{\mathcal D}_{0,\Phi\bar\Phi}(p) = \frac{1}{ip_5 + p^2/a'}, \qquad \hat{\mathcal D}_{0,\lambda\bar\lambda}(p) = \hat{\mathcal D}_{0,\mu\bar\mu}(p) = \frac{1}{ip_5 + p^2/a}. \tag{4.4}$$

The 2×2 block is also straightforward to invert (after using integration by parts to generate a lower left term). The non-vanishing transverse free propagators are then

$$\hat{\mathcal D}^{\rm tr}_{0,A^\mu A^\nu}(p) = \frac{2P^{\rm tr}_{\mu\nu}}{p_5^2 + (p^2)^2}, \qquad \hat{\mathcal D}^{\rm tr}_{0,A^\mu\Pi^\nu}(p) = -\hat{\mathcal D}^{\rm tr}_{0,\Pi^\mu A^\nu}(p) = \frac{2P^{\rm tr}_{\mu\nu}}{ip_5 + p^2}. \tag{4.5}$$

Using the Hodge decomposition for vector fields

$$X_\mu = X^{\rm tr}_\mu + \partial_\mu F \tag{4.6}$$

the longitudinal piece is easily inverted as well. The result is

$$\hat{\mathcal D}^{\rm lg}_{0,A^\mu A^\nu}(p) = \frac{2P^{\rm lg}_{\mu\nu}}{p_5^2 + (p^2)^2/a^2}, \quad \hat{\mathcal D}^{\rm lg}_{0,A^\mu\Pi^\nu}(p) = -\hat{\mathcal D}^{\rm lg}_{0,\Pi^\mu A^\nu}(p) = \frac{2P^{\rm lg}_{\mu\nu}}{ip_5 + p^2/a}, \quad \hat{\mathcal D}_{0,\Pi^\mu\Pi^\nu} = 0\,. \tag{4.7}$$

Note that if p_5 is integrated out in $\hat{\mathcal D}_{0,A^\mu A^\nu}$, the correct 4-$d$ Yang-Mills propagator is recovered. The (Ψ, c) 2×2 block similarly gives the following non-vanishing free propagators

$$\hat{\mathcal D}^{\rm tr}_{0,\Psi^\mu\bar\Psi^\nu}(p) = \frac{\delta_{\mu\nu}}{ip_5 + p^2} \qquad \hat{\mathcal D}^{\rm lg}_{0,\Psi^\mu\bar\Psi^\nu}(p) = \frac{\delta_{\mu\nu}}{ip_5 + p^2/a} \tag{4.8}$$

$$\hat{\mathcal D}^{\rm lg}_{0,c\bar\Psi^\nu}(p) = \frac{a'-a}{aa'}\,\frac{p_\nu}{(ip_5 + p^2/a)(ip_5 + p^2/a')} \qquad \hat{\mathcal D}_{0,c\bar c}(p) = \frac{1}{ip_5 + p^2/a'}. \tag{4.9}$$

Upon taking the inverse Fourier transform

$$\mathcal D_0(t,x) = \frac{1}{(2\pi)^5}\int dp_5\, e^{itp_5}\int d^4p\, e^{ix\cdot p}\hat{\mathcal D}_0(p_5,p) \tag{4.10}$$

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one sees that all the *free* propagators for the ghosts (and ghosts of ghosts) are retarded, since there is no pole in the upper p_5 half-plane (a and a' are positive) and closing the p_5 contour in the lower half-plane gives $\theta(t)$.

What about the complete propagators $\mathcal{D}$ then? For motivation consider the Green's functions G_0 and G (for $\lambda, \bar{\lambda}$ say), satisfying

$$\begin{aligned} (\partial_t - a^{-1}\partial^\mu\partial_\mu)G_0(t-s;x-y) &= \delta(s,y) \\ (\partial_t - a^{-1}D^\mu\partial_\mu)G(t-s;x,y;A) &= \delta(s,y) \end{aligned} \tag{4.11}$$

Of course $G_0 = \mathcal{D}_0$ is just the free propagator, but $G \neq \mathcal{D} = \langle T\lambda(t,x)\bar{\lambda}(s,y)\rangle_{I^{\min}}$ since the (time ordered) correlator involves integration over $\mathscr{D}A$ as well. Nevertheless, it is instructive to look at properties of G prior to integration. From Duhamel's principle [14] we have the following convolution relation between G and G_0.

$$G(t-s;x,y;A) = G_0(t-s;x-y) + \int_s^t d\tau \int d\xi\, G_0(t-\tau;x-\xi)[A^\mu(\xi), \partial_\mu G(\tau;\xi,A)] \tag{4.12}$$

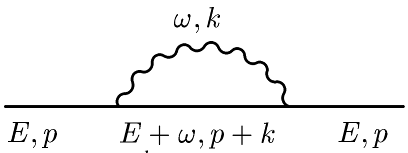
where we have suppressed indices. From this we see that not only is G automatically retarded as well, but for a sufficiently regular A one would conclude that

$$\lim_{t\searrow s} G(t-s;x,y;A) = \lim_{t\searrow s} G_0(t-s;x-y) = \delta(x-y) \tag{4.13}$$

The significance is that canonical commutation relations are *formally* satisfied if one assumes regularity of A and

$$\lim_{t\searrow s}\mathcal{D}(t-s;x,y) \equiv \lim_{t\searrow s}\langle G(t-s;x,y;A)\rangle = \langle \lim_{t\searrow s} G(t-s;x,y;A)\rangle \tag{4.14}$$

However, one cannot assume this so a separate argument is required to check the divergent case. We now proceed to show that the conclusion indeed applies to $\mathcal{D}$ as well, for $d < 4$ by dimensional regularization at one-loop level. This amounts to evaluating diagrams like this one, which is the first-order correction to the free $\Psi\bar{\Psi}$ propagator

$\hat{\Sigma}(E,p) =$  (4.15)

$$= \frac{1}{a^2}\int \frac{d\omega\, d^dk}{(2\pi)^{d+1}}\, p^\mu \left[\frac{2}{\omega^2 + (k^2)^2}\hat{P}^{\rm tr}_{\mu\nu}(k) + \frac{2}{\omega^2 + (k^2)^2/a^2}\hat{P}^{\rm lg}_{\mu\nu}(k)\right] \times$$
$$\times \frac{(p+k)^\nu}{(iE+\omega) + (p+k)^2/a} \tag{4.16}$$

with

$$\hat{\mathcal{D}} = \hat{\mathcal{D}}_0 + \hat{\mathcal{D}}_0\hat{\Sigma}\hat{\mathcal{D}}_0 + \cdots = \hat{\mathcal{D}}_0 + \hat{\mathcal{D}}_1 + \cdots . \tag{4.17}$$

Let us outline the computation of $\hat{\mathcal{D}}_1^{\rm tr}(E,p)$, the evaluation of the longitudinal component being similar. Since this is a standard but lengthy diagram integral regularization, for reasons of continuity we omit full detail here, which may be found in appendix B.

Expressing the denominators as parameter integrals

$$\frac{1}{B^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\alpha \, \alpha^{z-1} e^{-B\alpha} \tag{4.18}$$

one obtains after some computation the following expression

$$\hat{\mathcal{D}}_1^{\rm tr}(t,p) = \Gamma(\epsilon) \frac{(d-1)p^2}{d(16\pi)^{d/2}} \int_0^{\frac{1}{1+a}} \frac{d\alpha}{\alpha^\epsilon} [1-(1+a)\alpha] \int \frac{dE}{2\pi} \frac{e^{iEt}}{[(1-\alpha)p^2 + iEa]^\epsilon (iE + p^2/a)^2} \tag{4.19}$$

where $2\epsilon \equiv 4 - d$ and we have taken the inverse Fourier transform in E. Using parameter integrals again and integrating gives

$$\frac{d-1}{(1+a)d(16\pi)^{d/2}} \left(\frac{1+a}{a} t\right)^\epsilon e^{-p^2 t/a} \left(\frac{p^2 t}{a}\right) \int_0^1 \frac{d\alpha}{\alpha^\epsilon}(1-\alpha) \int_0^1 \frac{dx}{x} x^\epsilon (1-x) e^{-\alpha \frac{p^2 t}{a(1+a)} x} \tag{4.20}$$

$\Gamma(\epsilon)$ being cancelled by writing $[(1-\alpha)p^2 + iEa]^\epsilon$ as a parameter integral. Expanding, we find

$$\hat{\mathcal{D}}_1^{\rm tr}(t,p) = \frac{3/2}{(32\pi)^2} e^{-p^2 t/a} \frac{p^2 t}{a(1+a)} \left[\frac{1}{\epsilon} + 1 - \log \frac{t}{a(1+a)} + O(\epsilon \log t)\right] \tag{4.21}$$

and so $\lim_{t \searrow 0} \hat{\mathcal{D}}_1^{\rm tr}(t,p) = 0$.

Since (apart from the gauge parameters a and a') all the ghost propagators except $\hat{\mathcal{D}}_{0,c\bar{\Psi}}$ are of the same form it is clear that these considerations also apply to them. And if one rewrites $\mathcal{D}^{\rm lg}_{0,c\bar{\Psi}^\nu}$ as

$$\frac{p_\nu}{p^2} \left(\frac{1}{ip_5 + p^2/a'} - \frac{1}{ip_5 + p^2/a}\right) \tag{4.22}$$

it is not difficult to see that the result will be true here as well. Thus our conclusion applies to all ghost propagators.

5. Ground state

We now turn to the *ground state* P of the theory. It is the zero eigenvector of the hamiltonian

$$\hat{H} P = 0 \tag{5.1}$$

normalized to be a probability density, $\int \mathscr{D}M \, P = 1$, where $\mathscr{D}M$ is a functional measure determined below and $\hat{H}$ is an appropriate operator form of $H^{\rm min}$. In this section we will show that P is just the ground state of the Fokker-Planck hamiltonian. Also, by showing that the equal time limit of correlators of the 5-d theory agrees with the corresponding expectation value with respect to P, we will use this to establish equivalence between the hamiltonian and bulk lagrangian quantization.

We take the operator representation of the fields to be

$$\Pi^\mu(x) = \frac{\delta}{\delta A_\mu(x)}, \qquad \Psi^\mu(x) = \frac{\delta}{\delta \bar{\Psi}_\mu(x)}, \qquad \Phi(x) = \frac{\delta}{\delta \bar{\Phi}(x)}, \qquad \text{etc.} \tag{5.2}$$

with all *ghosts* acting as functional derivatives with respect to the corresponding *antighosts* (similarly for ghosts of ghosts). Due to signs, we must choose $\Phi(x) = -\delta/\delta\bar{\Phi}(x)$ and $\mu(x) = -\delta/\delta\bar{\mu}(x)$. This achieves the correspondence between the graded operator commutator and graded bracket via

$$[\![\, , \,]\!] = -\{\ ,\ \} \tag{5.3}$$

Recall that $[\xi,\eta]^a = f^a_{\ bc}\xi^b\eta^c$ still denotes the Lie algebra (not operator) commutator.

In this representation correlators of operators $\mathcal{O}$ are computed by integrating over $\mathscr{D}M$ with weight P, where $\mathscr{D}M$ is the measure obtained after integrating out all ghosts in

$$\int \mathscr{D}A\,\mathscr{D}\Pi\,\mathscr{D}\Phi\mathscr{D}\bar{\Phi}\,\mathscr{D}\mu\mathscr{D}\bar{\mu}\,\mathscr{D}\Psi\mathscr{D}\bar{\Psi}\,\mathscr{D}c\mathscr{D}\bar{c}\,\mathscr{D}\lambda\mathscr{D}\bar{\lambda}\,\exp(I^{\min}) \tag{5.4}$$

That the result is of the form

$$\begin{aligned}\mathscr{D}M &= \mathscr{D}A\Sigma(A)\,\mathscr{D}\bar{\Psi}\bar{\Psi}\,\mathscr{D}\bar{\lambda}\bar{\lambda}\,\mathscr{D}\bar{c}\bar{c}\,\mathscr{D}\bar{\Phi}\delta(\bar{\Phi})\,\mathscr{D}\bar{\mu}\delta(\bar{\mu}) \\ &= \mathscr{D}A\Sigma(A)\,\mathscr{D}\bar{\Psi}\delta(\bar{\Psi})\,\mathscr{D}\bar{\lambda}\delta(\bar{\lambda})\,\mathscr{D}\bar{c}\delta(\bar{c})\,\mathscr{D}\bar{\Phi}\delta(\bar{\Phi})\,\mathscr{D}\bar{\mu}\delta(\bar{\mu})\end{aligned} \tag{5.5}$$

can be seen as follows. (Note that the second equality is simply the fact that in fermionic calculus $d\theta\,\theta = d\theta\,\delta(\theta)$.) First, integrate out μ. This contributes

$$\det{}^{-1}\Big(\partial_5 - D\cdot\partial\Big)\mathscr{D}\bar{\mu}\delta(\bar{\mu}) \tag{5.6}$$

and therefore kills the cubic term $\frac{1}{a}\bar{\mu}[\Psi^\nu,\partial_\nu\lambda]$. Then integrate out λ which contributes

$$\det\Big(\partial_5 - D\cdot\partial\Big)\mathscr{D}\bar{\lambda}\delta(\bar{\lambda}) \tag{5.7}$$

The determinants cancel, the $\lambda,\bar{\lambda},\mu,\bar{\mu}$ quartet dependence is gone from $I^{\min}$, and the contribution to $\mathscr{D}M$ is

$$\mathscr{D}\bar{\lambda}\bar{\lambda}\,\mathscr{D}\bar{\mu}\,\delta(\bar{\mu})\,. \tag{5.8}$$

Now for the rest of the measure, as $I^{\min}$ is quadratic in Π, one can integrate Π out to obtain

$$\int \mathscr{D}A\,\mathscr{D}\Phi\mathscr{D}\bar{\Phi}\,\mathscr{D}\Psi\mathscr{D}\bar{\Psi}\,\mathscr{D}c\mathscr{D}\bar{c}\,\exp\Big(I^{\min\prime}_{(0)} + I^{\min\prime}_{\rm int} + I_{\rm red}\Big) \tag{5.9}$$

where

$$I_{\rm red} = -\int d^5x\left(\partial_t A_\mu - \frac{1}{a}D_\mu\partial\cdot A - D^\lambda F_{\lambda\mu}\right)^2 \tag{5.10}$$

and similarly to (4.1) one has a quadratic form

$$I^{\min\prime}_{(0)} = -\int d^5x\,\big(\bar{\Psi}^\mu,\bar{c},\bar{\Phi}\big)\begin{pmatrix}\mathcal{A}_{\mu\nu} & \mathcal{B}_\mu & 0\\ \mathcal{C}_\nu & \mathcal{D} & 0\\ 0 & 0 & \mathcal{D}\end{pmatrix}\begin{pmatrix}\Psi^\nu\\ c\\ \Phi\end{pmatrix} \tag{5.11}$$

but with the full A-dependence kept in the operators

$$\mathcal{A}_{\mu\nu} = \delta_{\mu\nu}\partial_5 - \frac{1}{a}D_\mu\partial_\nu + \frac{1}{a}\delta_{\mu\nu} - [F_{\mu\nu},\,\cdot\,] - \delta_{\mu\nu}D^2 + D_\nu D_\mu$$

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$$
\begin{aligned}
\mathcal{B}_\mu &= 2\left[\dot{A}_\mu - D^\lambda F_{\lambda\mu}, \cdot\right] \\
\mathcal{C}_\nu &= \frac{1}{a}\partial_\nu - \frac{1}{a'}D_\nu \\
\mathcal{D} &= \partial_5 - \frac{1}{a'}D^2 + \frac{1}{a}[\partial\cdot A, \cdot]
\end{aligned}
\tag{5.12}
$$

and the cubic and quartic ghost interactions are collected in

$$
\begin{aligned}
I_{\text{int}}^{\text{min}\prime} = \int d^5x \Big(&-\frac{1}{a}\bar{\Phi}[\partial\cdot\Psi, c] + \frac{1}{a'}\bar{\Phi}[D_\mu\Psi^\mu, c] - \frac{1}{a'}\bar{\Phi}[\Psi^\mu, \Psi_\mu] + \\
&+ [\bar{\Psi}_\mu, \bar{\Psi}^\mu]\Phi + \frac{2}{a}\bar{\Phi}[\Psi^\mu, D_\mu c] - [\bar{\Psi}^\mu, c][\bar{\Psi}_\mu, c]\Big)\,.
\end{aligned}
\tag{5.13}
$$

Now observe that

$$
\begin{pmatrix} \mathcal{A}_{\mu\nu} & \mathcal{B}_\mu & 0 \\ \mathcal{C}_\nu & \mathcal{D} & 0 \\ 0 & 0 & \mathcal{D} \end{pmatrix} = \frac{\partial}{\partial t} - L_0(\partial) + L_{\text{int}}(A) \tag{5.14}
$$

is a perturbation of the parabolic operator $\partial/\partial t - L_0$ by $L_{\text{int}}(A)$. Let's ignore the ghost interactions $I_{\text{int}}^{\text{min}\prime}$ for the moment. Since $G_0 \equiv (\partial/\partial t - L_0)^{-1}$ is retarded one can use the arguments of [3] to expand

$$
\begin{aligned}
\det\left(\frac{\partial}{\partial t} - L_0 + L_{\text{int}}(A)\right) &= \text{const}\cdot\exp\operatorname{Tr}\log(\mathbf{1} + G_0 L_{\text{int}}(A)) \\
&= \text{const}\cdot\exp\operatorname{Tr}\left(L_{\text{int}}G_0 - \frac{1}{2}L_{\text{int}}G_0L_{\text{int}}G_0 + \cdots\right)
\end{aligned}
\tag{5.15}
$$

and only the A-dependent 'tadpole' $\exp\operatorname{Tr}(L_{\text{int}}G_0)$ survives. One sees that integrating out Ψ, c, Φ will generate just delta functions of $\bar{\Psi}, \bar{c}, \bar{\Phi}$ times the A-dependent tadpole term. Including the ghost interactions into L_{int} gives a ghost-dependent tadpole, which we ignore.

The upshot is that indeed, after including the $\bar{\lambda}, \bar{\mu}$ component, the measure must be of form (5.5), where $\Sigma(A)$ involves $I_{(0)}^{\text{min}\prime}$ (and tadpoles). This is not surprising for a measure that ought to give non-trivial results for quantities with zero ghost number. We shall not attempt to investigate $\Sigma(A)$ in detail, but we expect that after restriction to a time slice it will generate $P(A)$.

Since $gh(\hat{H}) = gh(H^{\text{min}}) = 0$ the hamiltonian preserves ghost number. (The actual form, i.e., operator ordering, of $\hat{H}$ will be dealt with below and is irrelevant for now.) Thus, when P is expanded in ghost degree

$$
P = \sum_j \overset{(j)}{P}, \qquad gh(\overset{(j)}{P}) = j \tag{5.16}
$$

each component solves the equation separately

$$
\hat{H}\overset{(j)}{P} = 0\,. \tag{5.17}
$$

But because the measure $\mathscr{D}M$ kills terms with non-zero ghost number one has

$$\int \mathscr{D}M\, P = \int \mathscr{D}M \overset{(0)}{P} \tag{5.18}$$

so only $\overset{(0)}{P}$ contributes and we impose the condition that the ground state is *independent of the antighosts*

$$P = \overset{(0)}{P} = \overset{(0)}{P}(A) \tag{5.19}$$

in agreement with the remark above regarding Σ and P. Also, since P only depends on A it follows that it must satisfy

$$\hat{Q}P = \hat{\Omega}P = 0 \tag{5.20}$$

where $\hat{Q}$ and $\hat{\Omega}$ are obtained from (3.1) and (3.3) by replacing all ghosts and Π by derivatives.

We now come to operator ordering. Propagators are time ordered correlators, so for any fields Ψ_j one has

$$\langle T\Psi_1(x)\Psi_2(y)\rangle = \theta(x_5 - y_5)\langle\Psi_1(x)\Psi_2(y)\rangle + (-)^{gh(\Psi_1)gh(\Psi_2)}\theta(y_5 - x_5)\langle\Psi_2(y)\Psi_1(x)\rangle\,. \tag{5.21}$$

But we have determined that the full ghost propagators are retarded, i.e., proportional to the θ function, so one has for the λ-$\bar{\lambda}$ propagator, for example

$$\begin{aligned}\theta(x_5 - y_5) \sim \langle T\lambda(x)\bar{\lambda}(y)\rangle \;&=\; \theta(x_5 - y_5)\langle\lambda(x)\bar{\lambda}(y)\rangle - \theta(y_5 - x_5)\langle\bar{\lambda}(y)\lambda(x)\rangle \\ \Rightarrow \quad &\langle\bar{\lambda}(y)\lambda(x)\rangle = 0\end{aligned} \tag{5.22}$$

and similarly for the other ghosts (and ghosts of ghosts). One then has agreement with the hamiltonian representation

$$\lim_{\delta t\to 0}\langle\lambda(t+\delta t, x)\bar{\lambda}(t, y)\rangle = \delta(x-y) = \int \mathscr{D}A\cdots\mathscr{D}\bar{\lambda}\bar{\lambda}\frac{\delta}{\delta\bar{\lambda}(x)}\bar{\lambda}(y)P(A) \tag{5.23}$$

Similarly for the μ-$\bar{\mu}$ correlator one has

$$\lim_{\delta t\to 0}\langle\mu(t+\delta t, x)\bar{\mu}(t, y)\rangle = \delta(x-y) = \int \mathscr{D}A\cdots\mathscr{D}\bar{\mu}\delta(\bar{\mu})\frac{\delta}{\delta\bar{\mu}(x)}\bar{\mu}(y)P(A) \tag{5.24}$$

and so on. So the effective hamiltonian ordering prescription is the time ordering. All *fields* go to the *right*, that is all derivative operators to the left. One readily checks that this is consistent with the Π-A propagator as well.

Finally, noting that $\Pi(x) = \frac{\delta}{\delta A(x)}$ is ordered to the left in $\hat{H}$ we find that $P(A)$ solves the Fokker-Planck equation

$$\begin{aligned}\hat{H}_{\rm FP}(A)P(A) &\equiv -\int d^4x\,\frac{\delta}{\delta A^\mu(x)}\left[\frac{\delta}{\delta A_\mu(x)} - K^\mu(x;A)\right]P(A) = 0\,, \\ K^\mu(x;A) &\equiv D_\lambda F^{\lambda\mu}(x) + a^{-1}D^\mu\partial\cdot A(x) = \frac{\delta S_{\rm YM}}{\delta A_\mu(x)} + a^{-1}D^\mu\partial\cdot A(x)\end{aligned} \tag{5.25}$$

since $\hat{H}$ is effectively equal to precisely $\hat{H}_{\rm FP}$ when acting on functionals of A only. Of course the above considerations do not preclude degeneracy, i.e., we have not proven uniqueness here.

6. Conclusion

We treated the bulk-quantized gauge theory as a constrained gauge system and found that the canonical analysis of what happen to be particularly simple constraints leads directly to a BRST gauge-fixed hamiltonian and a corresponding action that agrees with (a reduced form of) the bulk action [5] arrived at in the lagrangian formulation. The hamiltonian is s-exact and w-closed

$$H^{\min} = -\{Q, \bar{Q}\} = -\{Q, X + \Omega f\}\,. \tag{6.1}$$

The lagrange multiplier fields for fixing the gauge were not included among the canonical variables (which we consider inconvenient due to a larger gauge algebra), but we made some elementary observations about how one may in principle proceed with inclusion of the lagrange multipliers in the phase space.

By dimensionally regularizing the self-energy one-loop correction to a representative ghost propagator we have concluded that the complete propagators for all ghosts (and ghosts of ghosts) are indeed retarded. Consequently we found that the ground state P depends on A only and is in fact just the ground state of the Fokker-Planck hamiltonian $P = P_{\mathrm{FP}}$. We have also displayed the consistency of the hamiltonian formulation at the quantum level in that expectation values with respect to the ground state P are compatible with the expectation values with respect to the 5-d action I.

Interesting questions to consider in the future may be renormalization of the equal-time theory governed by the Fokker-Planck equation and relation thereof to the renormalization of the 5-d theory governed by I.

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A. Nonminimal approach

In our opinion, the nonminimal treatment, which is based on a larger action

$$I_0^E = I_0 - \sum_{j=1}^{4} u^j \tilde{\varphi}_j \tag{A.1}$$

with more constraints and a larger phase space, does not lend itself to convenient quantization for the following reasons. As we shall see below, the gauge algebra contains 4 independent ϵ parameters (which we may reduce to 2 by hand). Therefore the BRST implementation of the nonminimal gauge symmetry would necessarily involve 2 extra conjugate ghost pairs (in addition to $\lambda, \bar{\lambda}$ and $\mu, \bar{\mu}$). Then one would need to find a proper gauge fixing fermion K that would give the same action as I (after integrating out the extra fields). Given the already imposing field content of the theory one would want to avoid bringing in more fields. In addition, the identification of all the ghosts associated with constraints with the ghosts in the lagrangian action $I_{\rm gf}$ may become tenuous since such identifications depend on the particular gauge fixing. For the minimal case the situation was quite simple regarding these issues, hence our choice. Nevertheless, in case there is further interest in the nonminimal direction, for completeness we include here a brief discussion of how one may approach the nonminimal treatment.

Since I_0 is already in first order form we can write

$$I_0 = \int d^5x \left(\dot{A}_\mu \Pi^\mu + \dot{\Psi}_\mu \bar{\Psi}^\mu + \dot{c}\bar{c} - \dot{\Phi}\bar{\Phi} - \mathcal{H}_C\right) \tag{A.2}$$

where $\mathcal{H}_C$ can be written out explicitly, but we don't need it now. We can immediately read off the canonical momenta

$$\Pi_{A_\mu} \equiv \frac{\delta I_0}{\delta \dot{A}_\mu} = \Pi^\mu\,, \qquad \Pi_{\Psi_\mu} \equiv \frac{\delta I_0}{\delta \dot{\Psi}_\mu} = \bar{\Psi}^\mu\,, \qquad \Pi_c \equiv \frac{\delta I_0}{\delta \dot{c}} = \bar{c}\,, \qquad \Pi_\Phi \equiv \frac{\delta I_0}{\delta \dot{\Phi}} = \bar{\Phi} \tag{A.3}$$

so there is no need to introduce independent momenta for these fields. There are thus two vanishing momenta

$$\Pi_5 \equiv \frac{\delta I_0}{\delta \dot{A}_\mu} = 0\,, \qquad \bar{\Psi}_5 \equiv \frac{\delta I_0}{\delta \dot{\Psi}_\mu} = 0\,. \tag{A.4}$$

One can then add the kinetic terms for A_5 and Ψ_5 and constrain them to zero by means of lagrange multipliers to obtain

$$I_0' = \int d^5x \left(\dot{A}_\mu \Pi^\mu + \dot{A}_5 \Pi^5 + \dot{\Psi}_\mu \bar{\Psi}^\mu + \dot{\Psi}_5 \bar{\Psi}^5 + \dot{c}\bar{c} - \dot{\Phi}\bar{\Phi} - \mathcal{H}_C - u^1 \mathcal{G}_1 - u^2 \mathcal{G}_2\right) \tag{A.5}$$

where

$$\mathcal{G}_1 \equiv \bar{\Psi}_5 \quad \text{and} \quad \mathcal{G}_2 \equiv \Pi_5 = s\mathcal{G}_1 \tag{A.6}$$

are the primary constraints and the u^j are new lagrange multiplier fields.

Now, however, $\{H_C, \mathcal{G}_j\} \approx 0$ generates two secondary constraints

$$\mathcal{G}_3 \equiv D_\mu \bar{\Psi}^\mu + \left[\bar{\Phi}, c\right] - \bar{c} \quad \text{and} \quad \mathcal{G}_4 \equiv D_\mu \Pi^\mu + \left[\bar{\Psi}_\mu, \Psi^\mu\right] + [\bar{c}, c] + \left[\bar{\Phi}, \Phi\right] = s\mathcal{G}_3 \tag{A.7}$$

and these are precisely φ_j from the minimal treatment. So the constraints $\varphi_1 = \mathcal{G}_3$ and $\varphi_2 = \mathcal{G}_4$ are now secondary. We find the following Lie algebra

$$\begin{aligned}
\{\mathcal{G}_3^a(x), \mathcal{G}_4^b(y)\} &= \delta(x-y) f^{ab}{}_c \mathcal{G}_3^c(y) \\
\{\mathcal{G}_4^a(x), \mathcal{G}_4^b(y)\} &= \delta(x-y) f^{ab}{}_c \mathcal{G}_4^c(y) \\
\text{rest} &= 0
\end{aligned} \tag{A.8}$$

which is similar to (2.15). One finds $\{H_C, \mathcal{G}_3\} = \{H_C, \mathcal{G}_4\} = 0$, thus there are no tertiary constraints. As before, all the constraints are irreducible.

Before we take a look at how the fields actually transform under gauge transformations, we notice that the constraints G_m may be replaced by a new set of constraints, given by a linear combination of the old, $\tilde{\mathcal{G}}_m = L_m{}^n \mathcal{G}_n$, provided that the matrix L is invertible, so that the new $\tilde{\mathcal{G}}_m$ are still irreducible. Therefore it is permissible to define

$$\begin{aligned}
\tilde{\mathcal{G}}_1 &= \mathcal{G}_1\,, \qquad \tilde{\mathcal{G}}_3 \equiv \mathcal{G}_3 - \left[\bar{\Psi}_5, A_5\right], \\
\tilde{\mathcal{G}}_2 &= \mathcal{G}_2\,, \qquad \tilde{\mathcal{G}}_4 \equiv \mathcal{G}_4 + \left[\bar{\Psi}_5, \Psi_5\right] + [A_5, \Pi_5] = s\tilde{\varphi}_3\,.
\end{aligned} \tag{A.9}$$

The algebra of the $\tilde{\varphi}_m$ is slightly different from (A.8)

$$\begin{aligned}
&\{\tilde{\mathcal{G}}_1^a(x),\tilde{\mathcal{G}}_4^b(y)\} = \delta(x-y)f^{ab}{}_c\tilde{\mathcal{G}}_1^c(y) && \{\tilde{\mathcal{G}}_2^a(x),\tilde{\mathcal{G}}_4^b(y)\} = \delta(x-y)f^{ab}{}_c\tilde{\mathcal{G}}_2^c(y)\\
&\{\tilde{\mathcal{G}}_2^a(x),\tilde{\mathcal{G}}_3^b(y)\} = \delta(x-y)f^{ab}{}_c\tilde{\mathcal{G}}_1^c(y) && \{\tilde{\mathcal{G}}_3^a(x),\tilde{\mathcal{G}}_4^b(y)\} = \delta(x-y)f^{ab}{}_c\tilde{\mathcal{G}}_3^c(y)\\
&\{\tilde{\mathcal{G}}_4^a(x),\tilde{\mathcal{G}}_4^b(y)\} = \delta(x-y)f^{ab}{}_c\tilde{\mathcal{G}}_4^c(y) && \text{rest} = 0
\end{aligned} \tag{A.10}$$

Note again that the action of s on the left-hand column gives the right-hand column. The $\tilde{\mathcal{G}}_3$ and $\tilde{\mathcal{G}}_4$ generate new gauge transformations on the A_5 quartet, while the rest of the fields transform the same way. We list the non-trivial gauge transformations

$$\begin{aligned}
&\delta_{\epsilon_1}\Psi_5^a = \varepsilon_1^a && \delta_{\epsilon_4}A_\mu^a = -(D_\mu\varepsilon_4)^a && \tilde{\delta}_{\epsilon_3}\Psi_5^a = [\varepsilon_3,A_5]^a\\
&\delta_{\epsilon_2}A_5^a = \varepsilon_2^a && \delta_{\epsilon_4}\Psi_\mu^a = [\varepsilon_4,\Psi_\mu]^a && \tilde{\delta}_{\epsilon_3}\Pi_5^a = [\varepsilon_3,\bar{\Psi}_5]^a\\
&\delta_{\epsilon_3}\Pi_\mu^a = [\varepsilon_3,\bar{\Psi}_\mu]^a && \delta_{\epsilon_4}\bar{\Psi}_\mu^a = [\varepsilon_4,\bar{\Psi}_\mu]^a && \tilde{\delta}_{\epsilon_4}A_5^a = [\varepsilon_4,A_5]^a\\
&\delta_{\epsilon_3}\Psi_\mu^a = -(D_\mu\varepsilon_3)^a && \delta_{\epsilon_4}\Pi_\mu^a = [\varepsilon_4,\Pi_\mu]^a && \tilde{\delta}_{\epsilon_4}\Psi_5^a = [\varepsilon_4,\Psi_5]^a\\
&\delta_{\epsilon_3}\Phi^a = [\varepsilon_3,c]^a && \delta_{\epsilon_4}c^a = [\varepsilon_4,c]^a && \tilde{\delta}_{\epsilon_4}\bar{\Psi}_5^a = [\varepsilon_4,\bar{\Psi}_5]^a\\
&\delta_{\epsilon_3}c^a = -\varepsilon_3^a && \delta_{\epsilon_4}\bar{c}^a = [\varepsilon_4,\bar{c}]^a && \tilde{\delta}_{\epsilon_4}\Pi_5^a = [\varepsilon_4,\Pi_5]^a\\
&\delta_{\epsilon_3}\bar{c}^a = [\varepsilon_3,\bar{\Phi}]^a && \delta_{\epsilon_4}\Phi^a = [\varepsilon_4,\Phi]^a &&\\
& && \delta_{\epsilon_4}\bar{\Phi}^a = [\varepsilon_4,\bar{\Phi}]^a &&
\end{aligned} \tag{A.11}$$

If one sets $\varepsilon_1 = -\dot{\varepsilon}_3$ and $\varepsilon_2 = -\dot{\varepsilon}_4$ one finds the full gauge transformations to be

$$\begin{aligned}
&\delta A_\nu = -D_\nu\varepsilon_4 && \delta\Psi_\nu = [\varepsilon_4,\Psi_\nu] - D_\nu\varepsilon_3 && \delta\bar{\Psi}_\nu = [\varepsilon_4,\bar{\Psi}_\nu] && \delta\Pi_\nu = [\varepsilon_4,\Pi_\nu] + [\varepsilon_3,\bar{\Psi}_\nu]\\
&\delta A_5 = -D_5\varepsilon_4 && \delta\Psi_5 = [\varepsilon_4,\Psi_5] - D_5\varepsilon_3 && \delta\bar{\Psi}_5 = [\varepsilon_4,\bar{\Psi}_5] && \delta\Pi_5 = [\varepsilon_4,\Pi_5] + [\varepsilon_3,\bar{\Psi}_5]\\
&\delta c = [\varepsilon_4,c] - \varepsilon_3 && \delta\Phi = [\varepsilon_4,\Phi] + [\varepsilon_3,c] && \delta\bar{\Phi} = [\varepsilon_4,\bar{\Phi}] && \delta\bar{c} = [\varepsilon_4,\bar{c},] + [\varepsilon_3,\bar{\Phi}]
\end{aligned} \tag{A.12}$$

which again agrees with the corresponding part of the w algebra (1.3) if the infinitesimal ε gauge parameters are replaced by λ and μ.

B. Propagator correction

In this section we work out in detail the evaluation of the (transverse) one-loop propagator correction. The amputated diagram is given by the integral

$$\begin{aligned}
\hat{\Sigma}^{\rm tr}(E,p) &= \frac{2}{a^2}\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\,\frac{p^\mu}{\omega^2+(k^2)^2}\Big(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\Big)\frac{(p+k)^\nu}{i(E+\omega)+(p+k)^2/a}\\
&= \frac{2}{a^2}\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\,\big(\delta_{\mu\nu}k^2-k_\mu k_\nu\big)p^\mu(p+k)^\nu\,\times\\
&\quad\times\frac{1}{k^2}\,\frac{1}{i\omega+k^2}\,\frac{1}{-i\omega+k^2}\,\frac{1}{i(E+\omega)+(p+k)^2/a}\,.
\end{aligned} \tag{B.1}$$

Using the parameter integral (4.18) this is expressed as

$$\begin{aligned}
&\frac{2}{a^2}\int\frac{d^dk\,d\omega}{(2\pi)^{d+1}}\,\big[p^2k^2-(p\cdot k)^2\big]\int_0^\infty d\alpha\,e^{-[i(E+\omega)+(p+k)^2/a]\alpha}\,\times\\
&\quad\times\int_0^\infty d\beta\,e^{-(k^2+i\omega)\beta}\int_0^\infty d\gamma\,e^{-(k^2-i\omega)\gamma}\int_0^\infty d\lambda\,e^{-k^2\lambda}
\end{aligned}$$

which after carrying out the $d\omega$ integration yields

$$\frac{2}{a^2}\int\frac{d^dk}{(2\pi)^d}\left[p^2k^2-(p\cdot k)^2\right]\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda\times$$
$$\times\int_0^\infty d\gamma\,\delta(\alpha+\beta-\gamma)\exp-\left[\left(\frac{(p+k)^2}{a}-iE\right)\alpha+k^2(\beta+\gamma+\lambda)\right]=$$
$$=\frac{2}{a^2}\int\frac{d^dk}{(2\pi)^d}[p^2k^2-(p\cdot k)^2]\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda\times$$
$$\times\exp-\left[\frac{(k^2+2k\cdot p+p^2)\alpha}{a}-iE\alpha+k^2(\lambda+\alpha+2\beta)\right]. \tag{B.2}$$

Changing variables $\alpha\to\alpha/a$, $\beta\to\beta/2$ gives

$$\int\frac{d^dk}{(2\pi)^d}\left[p^2k^2-(p\cdot k)^2\right]\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda\exp-\left[\xi k^2+2(k\cdot p)\alpha+p^2\alpha-iEa\alpha\right] \tag{B.3}$$

where we defined $\xi=\alpha+a(\alpha+\beta)+\lambda$. Shifting $k\to k+\alpha p/\xi$ and noting that $p^2k^2-(p\cdot k)^2$ is translation-invariant, this becomes

$$\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda e^{-\alpha\left(\frac{\xi-\alpha}{\xi}p^2+iEa\right)}\int\frac{d^dk}{(2\pi)^d}\left[p^2k^2-(p\cdot k)^2\right]e^{-\xi k^2}. \tag{B.4}$$

Performing the d^dk integration (we take the symmetric limit $k^\mu k^\nu\to\delta^{\mu\nu}k^2/d$) gives

$$\frac{(d-1)p^2}{d(16\pi)^{d/2}}\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda\frac{1}{\xi^{d/2+1}}e^{-\alpha\left(\frac{\xi-\alpha}{\xi}p^2+iEa\right)}. \tag{B.5}$$

We next insert $1=\int_0^\infty d\eta\,\delta(\eta-\xi)$ into the integral and change variables

$$\alpha\to\eta\alpha\,,\quad\beta\to\eta\beta\,,\quad\lambda\to\eta\lambda\qquad\Rightarrow\qquad\xi\to\eta\xi \tag{B.6}$$

noting that the delta function transforms as $\delta(\eta-\xi)\to\delta(\eta(1-\xi))=\delta(\xi-1)/\eta$ and the integral becomes

$$C(d)p^2\int_0^\infty d\alpha\int_0^\infty d\beta\int_0^\infty d\lambda\delta(\xi-1)\int_0^\infty\frac{d\eta}{\eta}\eta^{\frac{4-d}{2}}e^{-\alpha[(1-\alpha)p^2+iEa]\eta}. \tag{B.7}$$

We have defined

$$C(d)\equiv\frac{d-1}{d(16\pi)^{d/2}} \tag{B.8}$$

to unburden the notation. Observing that the delta function $\delta(\xi-1)=\delta(\lambda+(1+a)\alpha+a\beta-1)$ effectively constrains the $d\alpha\,d\beta\,d\lambda$ integration to a 2-simplex, we integrate out $d\lambda$ to get

$$C(d)p^2\iint\limits_{\substack{0\leqslant(1+a)\alpha+a\beta\leqslant1\\\alpha,\beta\geqslant0}}d\alpha\,d\beta\int_0^\infty\frac{d\eta}{\eta}\eta^{\frac{4-d}{2}}e^{-\alpha[(1-\alpha)p^2+iEa]\eta}. \tag{B.9}$$

Integrating over $d\beta$ and $d\eta$ results in

$$\hat\Sigma^{\rm tr}(E,p)=\frac{C(d)}{a}p^2\int_0^{\frac{1}{1+a}}d\alpha[1-(1+a)\alpha]\frac{\Gamma(\epsilon)}{\alpha^\epsilon[(1-\alpha)p^2+iEa]^\epsilon} \tag{B.10}$$

where $\epsilon \equiv (4-d)/2$ is the dimensional regularization parameter. Attaching the legs onto $\hat{\Sigma}$ and taking the inverse Fourier transform in E gives the one loop correction

$$\begin{aligned}
\hat{\mathcal{D}}_1^{\mathrm{tr}}(t,p) &= (\hat{\mathcal{D}}_0 \hat{\Sigma}^{\mathrm{tr}} \hat{\mathcal{D}}_0)^{\vee}(t,p) \\
&= \int \frac{dE}{2\pi}\, e^{iEt}\, \hat{\mathcal{D}}_1^{\mathrm{tr}}(E,p) \qquad \text{(B.11)} \\
&= \Gamma(\epsilon) \frac{C(d)}{a} p^2 \int_0^{\frac{1}{1+a}} \frac{d\alpha}{\alpha^\epsilon} [1-(1+a)\alpha] \int \frac{dE}{2\pi} \frac{e^{iEt}}{[(1-\alpha)p^2 + iEa]^\epsilon (iE + p^2/a)^2}\,.
\end{aligned}$$

Scaling $E \to E/a$ leads to

$$\hat{\mathcal{D}}_1^{\mathrm{tr}}(t,p) = \Gamma(\epsilon) C(d) p^2 \int_0^{\frac{1}{1+a}} \frac{d\alpha}{\alpha^\epsilon} [1-(1+a)\alpha] \int \frac{dE}{2\pi} \frac{e^{iEt/a}}{[(1-\alpha)p^2 + iE]^\epsilon (p^2 + iE)^2} \qquad \text{(B.12)}$$

Let us now deal with the dE integral

$$\int \frac{dE}{2\pi} \frac{e^{iEt/a}}{[(1-\alpha)p^2 + iE]^\epsilon (p^2 + iE)^2}\,. \qquad \text{(B.13)}$$

Rewriting the denominator using the parameter integrals, this equals

$$\int \frac{dE}{2\pi} e^{iEt/a} \frac{1}{\Gamma(\epsilon)} \int_0^\infty \frac{dx}{x} x^\epsilon e^{-[(1-\alpha)p^2 + iE]x} \int_0^\infty dy\, y\, e^{-(p^2+iE)y}\,. \qquad \text{(B.14)}$$

Integrating out dE gives a δ-function so we have

$$\frac{1}{\Gamma(\epsilon)} \int_0^\infty \frac{dx}{x} x^\epsilon \int_0^\infty dy\, y\, \delta\left(x + y - \frac{t}{a}\right) e^{-[(1-\alpha)x + y]p^2}\,. \qquad \text{(B.15)}$$

Letting $x \to xt/a$, $y \to yt/a$ (and so $\delta(x+y-t/a) \to a\delta(x+y-1)/t)$ gives

$$\begin{aligned}
&\frac{1}{\Gamma(\epsilon)} \left(\frac{t}{a}\right)^{1+\epsilon} \int_0^\infty \frac{dx}{x}\, x^\epsilon \int_0^\infty dy\, y\, \delta(x+y-1) e^{-[(1-\alpha)x+y]p^2 t/a} = \\
&\qquad = \frac{1}{\Gamma(\epsilon)} \left(\frac{t}{a}\right)^{1+\epsilon} \int_0^1 \frac{dx}{x}\, x^\epsilon (1-x) e^{(\alpha x - 1)p^2 t/a} \qquad \text{(B.16)}
\end{aligned}$$

So we obtain the following expression for the one-loop correction (B.11) (note that $\Gamma(\epsilon)$ cancels)

$$\hat{\mathcal{D}}_1^{\mathrm{tr}}(t,p) = C(d) p^2 e^{-p^2 t/a} \left(\frac{t}{a}\right)^{1+\epsilon} \int_0^{\frac{1}{1+a}} \frac{d\alpha}{\alpha^\epsilon} (1-(1+a)\alpha) \int_0^1 \frac{dx}{x} x^\epsilon (1-x) e^{\alpha x p^2 t/a} \qquad \text{(B.17)}$$

which becomes

$$\frac{C(d)}{1+a} \left(\frac{1+a}{a} t\right)^\epsilon e^{-p^2 t/a} \left(\frac{p^2 t}{a}\right) \int_0^1 \frac{d\alpha}{\alpha^\epsilon} (1-\alpha) \int_0^1 \frac{dx}{x}\, x^\epsilon (1-x) e^{-\alpha \frac{p^2 t}{a(1+a)} x} \qquad \text{(B.18)}$$

after rescaling $\alpha \to \alpha/(1+a)$. Clearly this integral is finite for $\epsilon > 0$. We Taylor expand the integrand in t and integrate dx to find

$$\frac{C(d)}{1+a} \left(\frac{1+a}{a} t\right)^\epsilon e^{-p^2 t/a} \left(\frac{p^2 t}{a}\right) \int_0^1 \frac{d\alpha}{\alpha^\epsilon} (1-\alpha) \left[\frac{1}{\epsilon} - 1 + \frac{\alpha p^2 t}{2a(1+a)} + O(\epsilon)\right] \qquad \text{(B.19)}$$

The remainder $O(\epsilon)$ consists of terms constant and higher order in t. Evaluating the remaining $d\alpha$ integral we finally have

$$\begin{aligned}\hat{\mathcal{D}}_1^{\rm tr}(t,p) &= \frac{C(4)}{2(1+a)}\left(\frac{1+a}{a}\,t\right)^{\epsilon} e^{-p^2t/a}(p^2t/a)\left[\frac{1}{\epsilon}+1+\frac{p^2t}{6a(1+a)}+\cdots\right] \\ &= \frac{3/2}{(32\pi)^2}e^{-p^2t/a}\frac{p^2t}{a(1+a)}\left[\frac{1}{\epsilon}+1-\log\frac{t}{a(1+a)}+O(\epsilon\log t)\right] \qquad \text{(B.20)}\end{aligned}$$

which is proportional to t. Apparently the $1/\epsilon$ pole generates a time renormalization counterterm, but we will not address the details of renormalizing the theory here. What is important is that the correction vanishes as $t \searrow 0$.

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